

# TWO KINDS OF HOOK LENGTH FORMULAS FOR COMPLETE $m$ -ARY TREES

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**Abstract.** In this paper, we define two kinds of hook length for internal vertices of complete  $m$ -ary trees, and deduce their corresponding hook length formulas, which generalize the main results obtained by Du and Liu.

**Keywords:** Hook length formulas,  $m$ -ary trees

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## 1. INTRODUCTION

Postnikov's hook length formula [3] states that

$$\frac{n!}{2^n} \sum_T \prod_v \left(1 + \frac{1}{h_v}\right) = (n+1)^{n-1},$$

where the sum is over all unlabeled complete binary trees  $T$  with  $n$  internal vertices, the product is over all internal vertices  $v$  of  $T$ , and  $h_v$  is the “hook length” of  $v$  in  $T$ , namely, the number of internal vertices in the subtree of  $T$  rooted at  $v$ . Postnikov derived the formula indirectly and asked for a combinatorial proof which was provided by Seo [4], Chen and Yang [1]. Later, Lascoux conjectured that

$$\sum_T \prod_v \left(x + \frac{1}{h_v}\right) = \frac{1}{(n+1)!} \prod_{i=0}^{n-1} \left((n+1+i)x + n+1-i\right).$$

This is equivalent to the more suggestive form

$$(1.1) \quad \sum_T \prod_v \frac{(h_v+1)x - h_v + 1}{2h_v} = \frac{1}{n+1} \binom{(n+1)x}{n},$$

which was proved by Du and Liu [2]. Moreover, they generalized (1.1) from counting complete binary trees to counting complete  $(m+1)$ -ary trees and obtained the following formula for  $(m+1)$ -ary trees:

$$(1.2) \quad \sum_{T \in \mathcal{T}_{n,m+1}} \prod_v \frac{(mh_v+1)x - h_v + 1}{(m+1)h_v} = \frac{1}{mn+1} \binom{(mn+1)x}{n},$$

or equivalently

$$(1.3) \quad \sum_{T \in \mathcal{T}_{n,m+1}} \prod_v \left(x + \frac{1}{h_v}\right) = \frac{x+1}{n!} \prod_{i=1}^{n-1} \left((mn+i+1)(x+1) - (m+1)i\right).$$

where  $\mathcal{T}_{n,m+1}$  denotes the set of complete  $(m+1)$ -ary trees with  $n$  internal vertices, the product is over all internal vertices  $v$  of  $T$ .

Recall that a *plane forest* is a forest of plane trees that are linearly ordered. Let  $\mathcal{F}(n)$  denote the set of plane forests with  $n$  vertices. For any vertex  $v$  of  $F \in \mathcal{F}(n)$ , the hook length  $H_v$  of  $v$  is defined as the number of vertices in the subtree rooted at  $v$ . Note that this definition is slightly different to that of hook length defined above for  $(m+1)$ -ary trees. Du and Liu [2] investigated the hook length polynomials for plane forests and obtained that

$$(1.4) \quad \sum_{F \in \mathcal{F}(n)} \prod_{v \in V(F)} \left(x + \frac{1}{H_v}\right) = \frac{(x+1)}{n!} \prod_{i=1}^{n-1} \left((2n+1-i)(x+1) - i\right),$$

or equivalently,

$$(1.5) \quad F_n(x) = \sum_{F \in \mathcal{F}(n)} \prod_{v \in V(F)} \frac{(2h_v - 1)x - H_v + 1}{H_v} = \frac{1}{2n+1} \binom{(2n+1)x}{n},$$

where  $V(F)$  is the set of vertices of  $F$ .

It is well known that there exists a simple bijection between plane forests and complete binary trees. For the sake of completeness, we present it here. Given any plane forest  $F \in \mathcal{F}(n)$ , we pick the first plane tree  $T$  of  $F$  with root  $u$ . Let  $T'$  denote the plane forest deduced from  $T$  by removing the root  $u$ . Then the bijection can be defined recursively as follows:  $\psi(F)$  is the complete binary tree with root  $u$  such that it has the left subtree  $\psi(T')$  and the right subtree  $\psi(F \setminus T)$ .

It is clear that the bijection maps the hook length of  $v$  in  $V(F)$  to the number of internal vertices of the left component of  $v$  of  $\psi(F)$ . This motivates us to define the first kind of *hook length*  $\mathcal{H}_v$  for an internal vertex  $v$  of  $m$ -ary trees  $T$ . Let  $T_v$  denote the  $m$ -ary subtree of  $T$  rooted at  $v$  and let  $T'_v$  denote the reduced tree from  $T_v$  by removing the rightmost subtree of  $v$ . Define  $\mathcal{H}_v$  to be the number of internal vertices of the subtree  $T'_v$ . See Figure 1 for example. Note that in the case  $m = 2$ , the hook length  $\mathcal{H}_v$  reduces to  $H_v$  up to the bijection  $\psi$ . Then we have the first main result which is a generalization of (1.4) and (1.5).

**Theorem 1.1.** *For any integer  $m \geq 2$ ,*

$$(1.6) \quad \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \left(x + \frac{1}{\mathcal{H}_v}\right) = \frac{(x+1)}{n!} \prod_{i=1}^{n-1} \left((mn+1-i)(x+1) - (m-1)i\right),$$

or equivalently,

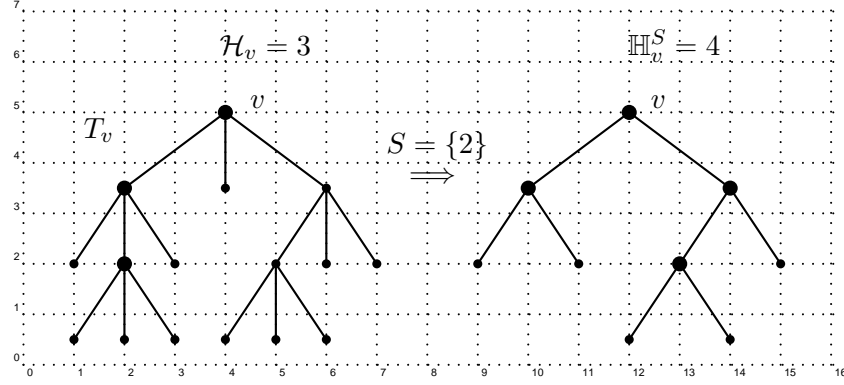
$$(1.7) \quad \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \frac{(m\mathcal{H}_v - 1)x - \mathcal{H}_v + 1}{(m-1)\mathcal{H}_v} = \frac{1}{mn+1} \binom{(mn+1)x}{n},$$

where  $\mathcal{I}(T)$  is the set of internal vertices of  $T \in \mathcal{T}_{n,m}$ .

Moreover, the definition of the first hook length inspires us defining the second kind of hook length. Let  $S$  be a subset of  $[m] = \{1, 2, \dots, m\}$ , for an internal vertex  $v$  of  $(m+1)$ -ary trees  $T$ , let  $T_v$  denote the  $(m+1)$ -ary subtree of  $T$  rooted at  $v$ , and let  $v_1, v_2, \dots, v_{m+1}$  be the children of  $v$ , first delete the subtree rooted at  $v_r$  for all  $r \in S$ , namely delete the  $r$ th subtree of  $v$  for all  $r \in S$ ; then delete the  $r$ th subtree of  $v_j$  for all  $r \in S$  and  $j \in [m+1] \setminus S$ , and then continue this process; one can obtain an  $(m+1-|S|)$ -ary tree  $T_v^S$ . Define  $\mathbb{H}_v^S$  to be the number of internal vertices of  $T_v^S$ . See Figure 1 for example. Then we have the second main result which, in the case  $S = \emptyset$ , reduces to (1.3) and (1.2) respectively.

**Theorem 1.2.** *For any integer  $m \geq 1$ ,  $S \subset [m]$  and  $s = |S|$ ,*

$$(1.8) \quad \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left(x + \frac{1}{\mathbb{H}_v^S}\right) = \frac{(x+1)}{n!} \prod_{i=1}^{n-1} \left((mn+i+1)(x+1) - (m-s+1)i\right),$$

FIGURE 1. The two kinds of hook length of  $v$ .

or equivalently,

$$(1.9) \quad \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \frac{((m-s)\mathbb{H}_v^S + 1)x - \mathbb{H}_v^S + 1}{(m-s+1)\mathbb{H}_v^S} = \frac{1}{mn+1} \binom{(mn+1)x}{n}.$$

In the next two Sections, we present the proofs of Theorem 1.1 and 1.2 respectively.

## 2. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we need the following lemma obtained by Seo [4].

**Lemma 2.1.** *Fix positive integers  $a$  and  $b$ . Let  $\Omega := \Omega(t) = 1 + \sum_{n \geq 1} \Omega_n t^n$  be a formal power series in  $t$  satisfying*

$$\Omega' = x\Omega^{b+1} + at\Omega^b\Omega',$$

where the prime denotes the derivative of  $\Omega$  with respect to  $t$ . Then  $\Omega_n$  can be given by

$$\Omega_n = \frac{x}{n!} \prod_{i=1}^{n-1} (ai + bx(n-i) + x).$$

*Proof of Theorem 1.1:* Define

$$\mathcal{H}_{n,m}(x) = \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \frac{(m\mathcal{H}_v - 1)x - \mathcal{H}_v + 1}{(m-1)\mathcal{H}_v}.$$

Given any  $m$ -ary tree  $T \in \mathcal{T}_{n,m}$  with root  $u$  for  $n \geq 1$ , let  $T_1, T_2, \dots, T_m$  be the  $m$  subtrees of  $u$  from left to right with  $i_1, i_2, \dots, i_m$  internal vertices respectively. Then  $\mathcal{H}_u = i_1 + i_2 + \dots + i_{m-1} + 1$ . Therefore, we can deduce the recurrence relation for  $\mathcal{H}_{n,m}(x)$ ,

$$\begin{aligned} \mathcal{H}_{n,m}(x) &= \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \frac{(m\mathcal{H}_v - 1)x - \mathcal{H}_v + 1}{(m-1)\mathcal{H}_v} \\ &= \sum_{i_1+i_2+\dots+i_m=n-1} \frac{(m\mathcal{H}_u - 1)x - \mathcal{H}_u + 1}{(m-1)\mathcal{H}_u} \prod_{j=1}^m \sum_{T_j \in \mathcal{T}_{i_j,m}} \prod_{v \in \mathcal{I}(T_j)} \frac{(m\mathcal{H}_v - 1)x - \mathcal{H}_v + 1}{(m-1)\mathcal{H}_v} \\ &= \sum_{i_1+i_2+\dots+i_m=n-1} \left( \frac{mx-1}{m-1} + \frac{1-x}{(m-1)(i_1+i_2+\dots+i_{m-1}+1)} \right) \prod_{j=1}^m \mathcal{H}_{i_j,m}(x) \end{aligned}$$

Define the generating function for  $\mathcal{H}_{n,m}(x)$  by

$$\mathcal{H}_m(x; t) = 1 + \sum_{n \geq 1} \mathcal{H}_{n,m}(x) t^n.$$

Then by the above relation and the following series expansion

$$\mathcal{H}_m^k(x; t) = 1 + \sum_{n \geq 1} t^n \sum_{i_1 + i_2 + \dots + i_k = n} \prod_{j=1}^k \mathcal{H}_{i_j, m}(x),$$

one can get

$$\mathcal{H}_m(x; t) = 1 + \frac{mx-1}{m-1} t \mathcal{H}_m^m(x; t) + \frac{1-x}{m-1} \mathcal{H}_m(x; t) \int_0^t \mathcal{H}_m^{m-1}(x; y) dy,$$

from which, one can derive that

$$\mathcal{H}_m'(x; t) = x \mathcal{H}_m^{m+1}(x; t) + (mx-1) t \mathcal{H}_m^m(x; t) \mathcal{H}_m'(x; t),$$

where the prime denotes the derivative of  $\mathcal{H}_m(x; t)$  with respect to  $t$ .

Using Lemma 2.1, we have

$$\begin{aligned} \mathcal{H}_{n,m}(x) &= \frac{x}{n!} \prod_{i=1}^{n-1} \left( (m(n-i)+1)x + (mx-1)i \right) \\ &= \frac{1}{mn+1} \binom{(mn+1)x}{n}, \end{aligned}$$

which proves (1.7).

Dividing by  $\left(\frac{1-x}{m-1}\right)^n$  on both sides of (1.7), and then replacing  $\frac{mx-1}{1-x}$  by  $x$ , one can get (1.6) immediately.  $\square$

If choose the special values 0 or  $-m$  for  $x$  in (1.6), we get the following identities

**Corollary 2.2.** *For any integer  $m \geq 2$ ,*

$$\begin{aligned} \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \frac{1}{\mathcal{H}_v} &= \frac{m^n}{mn+1} \binom{\frac{mn+1}{m}}{n}, \\ \sum_{T \in \mathcal{T}_{n,m}} \prod_{v \in \mathcal{I}(T)} \left( m - \frac{1}{\mathcal{H}_v} \right) &= \frac{(m-1)^n}{n!} (mn+1)^{n-1}. \end{aligned}$$

### 3. PROOF OF THEOREM 1.2

Define

$$s\mathbb{H}_{n,m}(x) = \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left( x + \frac{1}{\mathbb{H}_v^S} \right),$$

and define the generating function for  $s\mathbb{H}_{n,m}(x)$  by

$$s\mathbb{H}_m(x; t) = 1 + \sum_{n \geq 1} s\mathbb{H}_{n,m}(x) t^n.$$

First, we consider the case when  $S$  is the empty set  $\emptyset$ . Note that in this case, (1.8) and (1.9) reduce to the results (1.3) and (1.2) obtained by Du and Liu [2]. Given any  $(m+1)$ -ary tree  $T \in \mathcal{T}_{n,m+1}$  with root  $u$  for  $n \geq 1$ , let  $T_1, T_2, \dots, T_{m+1}$  be the  $m+1$  subtrees of  $u$  from left to

right with  $i_1, i_2, \dots, i_{m+1}$  internal vertices respectively. Then  $\mathbb{H}_u^\emptyset = i_1 + i_2 + \dots + i_{m+1} + 1$ . Therefore, we can deduce a recurrence relation for  $\emptyset\mathbb{H}_{n,m}(x)$ ,

$$\begin{aligned} \emptyset\mathbb{H}_{n,m}(x) &= \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left( x + \frac{1}{\mathbb{H}_v^\emptyset} \right) \\ &= \sum_{i_1+i_2+\dots+i_{m+1}=n-1} \left( x + \frac{1}{\mathbb{H}_u^\emptyset} \right) \prod_{j=1}^{m+1} \sum_{T_j \in \mathcal{T}_{i_j,m+1}} \prod_{v \in \mathcal{I}(T_j)} \left( x + \frac{1}{\mathbb{H}_v^\emptyset} \right) \\ &= \sum_{i_1+i_2+\dots+i_{m+1}=n-1} \left( x + \frac{1}{i_1 + i_2 + \dots + i_{m+1} + 1} \right) \prod_{j=1}^{m+1} \emptyset\mathbb{H}_{i_j,m+1}(x). \end{aligned}$$

Similar to the proof of Theorem 1.1, an equation for  $\emptyset\mathbb{H}_m(x; t)$  can be derived as

$$\emptyset\mathbb{H}_m(x; t) = 1 + xt \emptyset\mathbb{H}_m^{m+1}(x; t) + \int_0^t \emptyset\mathbb{H}_m^{m+1}(x; y) dy,$$

from which, one can get

$$(3.1) \quad \emptyset\mathbb{H}'_m(x; t) = (x+1) \emptyset\mathbb{H}_m^{m+1}(x; t) + (m+1)xt \emptyset\mathbb{H}_m^m(x; t) \emptyset\mathbb{H}'_m(x; t),$$

where the prime denotes the derivative of  $\emptyset\mathbb{H}_m(x; t)$  with respect to  $t$ .

For any complete  $(m+1)$ -ary tree  $T$  with  $k \geq 1$  internal vertices and an  $s$ -subset  $S \in [m]$ , according to the definition of the second kind of hook length,  $T$  can be uniquely partitioned into a complete  $(m-s+1)$ -ary tree with  $n$  internal vertices for some  $n \geq 1$  and an ordered forest of  $ns$  complete  $(m+1)$ -ary trees. Hence we get a recurrence relation for  $s\mathbb{H}_m(x; t)$ , namely

$$(3.2) \quad s\mathbb{H}_m(x; t) = 1 + \sum_{n \geq 1} \emptyset\mathbb{H}_{n,m-s}(x) t^n s\mathbb{H}_m^{ns}(x; t) = \emptyset\mathbb{H}_{m-s}(x; s\mathbb{H}_m^s(x; t)t).$$

Taking the derivative on both side of (3.2) with respect to  $t$ , using (3.1), we have

$$(3.3) \quad s\mathbb{H}'_m(x; t) = (x+1) s\mathbb{H}_m^{m+1}(x; t) + ((m-s+1)x + s(x+1))t s\mathbb{H}_m^m(x; t) s\mathbb{H}'_m(x; t).$$

Applying Lemma 2.1 to (3.3), one can obtain that

$$\begin{aligned} s\mathbb{H}_{n,m}(x) &= \frac{x+1}{n!} \prod_{i=1}^{n-1} \left( ((m-s+1)x + s(x+1))i + m(x+1)(n-i) + x+1 \right) \\ &= \frac{(x+1)}{n!} \prod_{i=1}^{n-1} \left( (mn+i+1)(x+1) - (m-s+1)i \right), \end{aligned}$$

which proves (1.8).

Dividing by  $((m-s+1) - (x+1))^n$  on both sides of (1.8), and then replacing  $\frac{x+1}{(m-s+1)-(x+1)}$  by  $x$ , one can get (1.9) immediately.  $\square$

If choose the special values  $m-s-1$  or  $m-s$  for  $x$  in (1.8), or choose  $s=m$  in (1.9), we get the following identities

**Corollary 3.1.** *For any integer  $m \geq 0$ ,  $S \subset [m]$  and  $s = |S|$ ,*

$$\begin{aligned} \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left( m - s - 1 + \frac{1}{\mathbb{H}_v^S} \right) &= \frac{1}{mn+1} \binom{(mn+1)(m-s)}{n}, \\ \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \left( m - s + \frac{1}{\mathbb{H}_v^S} \right) &= \frac{(m-s+1)^n (mn+1)^{n-1}}{n!}, \\ \sum_{T \in \mathcal{T}_{n,m+1}} \prod_{v \in \mathcal{I}(T)} \frac{x - \mathbb{H}_v^{[m]} + 1}{\mathbb{H}_v^{[m]}} &= \frac{1}{mn+1} \binom{(mn+1)x}{n}. \end{aligned}$$

**Remark 3.2.** *Motivated by Lemma 2.1 and the proof of Theorem 1.2, we can consider the function  $\Phi := \Phi(t) = \Omega(t\Phi^s(t))$ , where  $\Omega(t)$  is defined in Lemma 2.1 and  $\Phi = 1 + \sum_{n \geq 1} \Phi_n t^n$ . Then it is easy to derive that  $\Phi(t)$  satisfies the following differential equation*

$$\Phi' = x\Phi^{b+s+1} + (a + sx)t\Phi^{b+s}\Phi',$$

*from which, by Lemma 2.1, we can deduce the explicit expression for  $\Phi_n$ ,*

$$\Phi_n = \frac{x}{n!} \prod_{i=1}^{n-1} (ai + bx(n-i) + (sn+1)x).$$

*We wonder if there is any combinatorial explanation for the relation.*

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